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Rapid Note

Spectral problems for the discrete velocity models

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Abstract. We use the discrete velocity models to study the spectral problems related to the 1D plane wave propagation in monatomic gases which are fundamental in the rarefied gases dynamics and nonequilibrium statistical thermodynamics. The results show that 6- and 8-velocity models can only capture the propagation of diffusion mode (entropy wave) in the intermediate Knudsen number regime. 4-velocity model instead captures the propagation of sound mode quite well after the comparison with the continuum-mechanic results.

PACS. 43.35.+d Ultrasonics, quantum acoustics, and physical effects of sound – 47.45.-n Rarefied gas dynamics – 62.80.+f Ultrasonic relaxation

1 Introduction

The problem of ultrasound dispersion is special because, although it is two-dimensional (x and t), the variables can be separated and, at least at the outset, the problem can be treated without boundaries. It is clear that the classical theory of sound involves a number of approximations, the first of which is linearity. But there are fundamental limitations, very long wavelengths and low frequencies. There have been several investigations [1-3] concerned with the spectral problems of very high frequency plane sound waves. Their main concern is on the measurable (sound) mode of propagation. Initially researchers used continuum-mechanic approaches, *e.g.* Navier-Stokes and/or its-extended equations to study this kind of problem [4]. The results obtained can only be valid up to the continuum limit or small Knudsen number (lowly rarefied) cases. The propagation of the Navier-Stokes *diffusion mode* (or the second mode [5] of the Navier-Stokes theory of sound propagation), which has its wave number equal to its attenuation rate for the range of continuum-regime, other than the sound mode in rarefied gases, however, were also reported then [2,5]. The main associated troubles are, as far as the authors know, there were no measurements about the propagation of diffusion mode.

Later on, ultrasound propagation in highly rarefied monatomic gases, *i.e.*, gases in which the ratio of molecular collision to sound frequency is small, had been studied by using the linearized Boltzmann equation for the dispersion relation since early '60s [6-9]. Even there were measurements of forced ultrasound propagation in gases for comparison, but questions regarding boundary conditions [10] were avoided by most of these workers who did initial value problems in an unbounded gas rather than the semi-infinite problem to which measurements refer.

By using the Discrete Boltzmann approaches, however, only in late '80s did some researchers start to study the 1D ultrasound propagation by calculating the speed of sound [9] or the dispersion relations [11] in the large where the velocity of propagation of sound wave can be classically determined by looking for the properties of the solutions of the conservation equations referred to the Maxwellian state. Their results were still centered on the sound mode of the spectra.

In the discrete Boltzmann approach, the main idea is to consider that the particle velocities belong to a given finite set of velocity vectors. Only the velocity space is discretized, the space and time variables being continuous. The discrete velocity models [12] thus come. For a lattice gas, the space and time variables are also discretized.

In this Paper, the spectra of discrete velocity models for hard-sphere gases are our main concern. The verification of our approaches with the previous available measurements (propagation of forced sound-mode) has been done in [13], the argues about the differences between different discrete velocity models included. Here, we only consider the 1D propagation of plane wave by neglecting the complicated real boundary conditions or the effects of transmitter/receivers. The results will be the propagation of sound as well as diffusion modes.

2 Formulations

We assume that the gas is composed of identical particles of the same mass. The velocities of these particles are restricted to, *e.g.*: $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_p$, p is a finite positive integer. The discrete number density of particles are denoted by $N_i(\mathbf{r}, t)$ associated with the velocity $\mathbf{v}_i$ at point $\mathbf{r}$ and time t. Considering binary collision only, the model of discrete Boltzmann equation proposed in [12] is a system of $2n(=p)$ semilinear partial differential equations of the hyperbolic type:

$$\frac{\partial}{\partial t}N_i + \mathbf{v}_i \cdot \frac{\partial}{\partial \mathbf{x}}N_i = \frac{2cS}{n}\sum_{j=1}^{n} N_j N_{j+n} - N_i N_{i+n}, \quad i = 1, \cdots, 2n, \quad (1)$$

where $N_i = N_{i+2n}$ are unknown functions, and $\mathbf{v}_i = c(\cos[(i-1)\pi/n], \sin[(i-1)\pi/n])$; c is a reference velocity modulus, S is an effective collision cross-section.

Since passage of the sound wave causes a small departure from equilibrium (Maxwellian type) resulting in energy loss owing to internal friction and heat conduction, we linearize above equations around a uniform Maxwellian state (N_0) by setting $N_i(t, \mathbf{x}) = N_0\ (1+P_i(t, \mathbf{x}))$, where P_i is a small perturbation. After some manipulations we then have

$$\left[\frac{\partial^2}{\partial t^2} + c^2 \cos^2 \frac{(m-1)\pi}{n}\frac{\partial^2}{\partial x^2} + 4cSN_0\frac{\partial}{\partial t}\right] D_m = \frac{4cSN_0}{n}\sum_{k=1}^{n}\frac{\partial}{\partial t}D_k, \quad (2)$$

where $D_m = (P_m + P_{m+n})/2$, $m = 1, \cdots, n$, since $D_1 = D_m$ for $1 = m \pmod{2n}$. We are ready to look for the solutions in the form of plane wave $D_m = a_m \exp i(kx - \omega t)$, $(m = 1, \cdots, n)$, with $\omega = \omega(k)$. This is related to the dispersion relations of 1D forced ultrasound propagation of rarefied gases problem. So we have

$$\left(1 + ih - 2\lambda^2 \cos^2 \frac{(m-1)\pi}{n}\right) a_m - \frac{ih}{n}\sum_{k=1}^{n} a_k = 0, \quad m = 1, \cdots, n, \quad (3)$$

where $\lambda = kc/(\sqrt{2}\omega)$, $h = 4cSN_0/\omega \propto 1/\mathrm{Kn}$ is the rarefaction parameter of the gas; Kn is the Knudsen number which is defined as the ratio of the mean free path of gases to the wave length of ultrasound.

Let $a_m = \mathcal{C}/(1+ih-2\lambda^2\cos^2[(m-1)\pi/n])$, where $\mathcal{C}$ is an arbitrary, unknown constant, since we here only have interests in the eigenvalues of above relation. The eigenvalue problems for different $2n$-velocity model reduces to $F_n\ (\lambda) = 0$, or

$$1 - \frac{ih}{n}\sum_{m=1}^{n}\frac{1}{1 + ih - 2\lambda^2\cos^2\frac{(m-1)\pi}{n}} = 0. \quad (4)$$

We solve $n = 2$, 3, and 4 cases respectively, *i.e.*, 4-velocity, 6-velocity and 8-velocity cases. The corresponding eigenvalue equations become algebraic polynomial-form with the complex roots being the results of λs [14,15].

For 2×2-velocity model, we obtain $1 - (ih/2) \sum_{m=1}^{2}[1/(1 + ih - 2\lambda^2\cos^2(m-1)\pi/2)] = 0$.

Likewise, we have $6\lambda^4 - (15+22ih)\lambda^2 - 8h^2 + 14ih + 6 = 0$; $k_0\lambda^8 + k_1\lambda^6 + k_2\lambda^4 + k_3\lambda^2 + k_4 = 0$, with $k_0 = 4b$, $k_1 = -12b - 10\hat{c}$, $k_2 = 13b + 12\hat{c} + 9b\cdot\hat{c}$, $k_3 = -6b - 10b\cdot\hat{c} - 4h^2\cdot\hat{c}$, $k_4 = 3b\cdot\hat{c} + \hat{c}^2\cdot b + 3b\,h^2\cdot\hat{c}$, where $b = 1+h^2$, $\hat{c} = 1-ih$; for 2×3-velocity model, 2×4-velocity model, respectively.

3 Results and discussions

We can obtain the complex roots (λs) for the polynomial equations above. The roots are the values for the nondimensional dispersion (real part) and the attenuation or absorption (imaginary part), respectively. The spectra of $2 \times n$-velocity model for $n = 2, 3, 4$ look entirely different [16]. The results for different models have been put into Figure 1, with the Navier-Stokes (N-S) data [2] included for comparison. We can observe that 4-velocity model captures the propagating behavior of sound mode quite well [13] and 6- and 8-velocity model, instead, seems to capture only the propagating mode of diffusion [2,5,16-18]. There are perhaps basic limitations or assumptions in all these models: the higher molecular velocities are missing [11] when the results of sound-mode were compared to the measurements or continuous kinetic theories. The real and imaginary part of the diffusion mode are, for both (6- and 8-velocity) models, increasing with the increasing rarefaction parameter h especially near the continuum-limit. Meanwhile those of the diffusion mode from the Navier-Stokes approach increase linearly with h.

From a more modern point of view, dissipation of the sound wave arises fundamentally because of a necessary coupling of density and energy fluctuations induced by the disturbance. Within one mean free path or so of an oscillating boundary, a free molecular flow solution can probably be computed. The damping will quite likely turn out to be linear because the damping mechanism is the shift in phase of particles which hit the wall at different times. To conclude for the results of sound mode, it was observed that, whereas the Navier-Stokes approach provides a good modelling at low frequencies, it is definitely not adequate at high frequencies $h \leq 2$. Especially the zero dispersion (phase speed) as h approaches zero [6]. As the wavelength of sound is made significantly shorter, so that the effects of viscosity and the heat conduction are no longer small, the validity of Navier-Stokes approach itself becomes questionable. If there is no rarefaction effect ($h = 0$), we have only real roots for all the models [16]. Once $h \neq 0$, the imaginary part appears and the spectra diagram for each model looks entirely different. We can see that, for 4-velocity model, there is one branch starting from $h = 0.1$ then to the maximum imaginary-part location $h = 1.692$ where the attenuation reaches the peak and finally reaching the limit at the real axis (*i.e.*, attenuation disappears, $\lambda_{\mathrm{real}} = 1$). But, for 8-velocity model, this behavior disappears, instead, there is one *turn* (the locus starting from $h = 0.1$) at $h = 1.2$ where the dispersion reaches the minimum. The differences between different velocity models

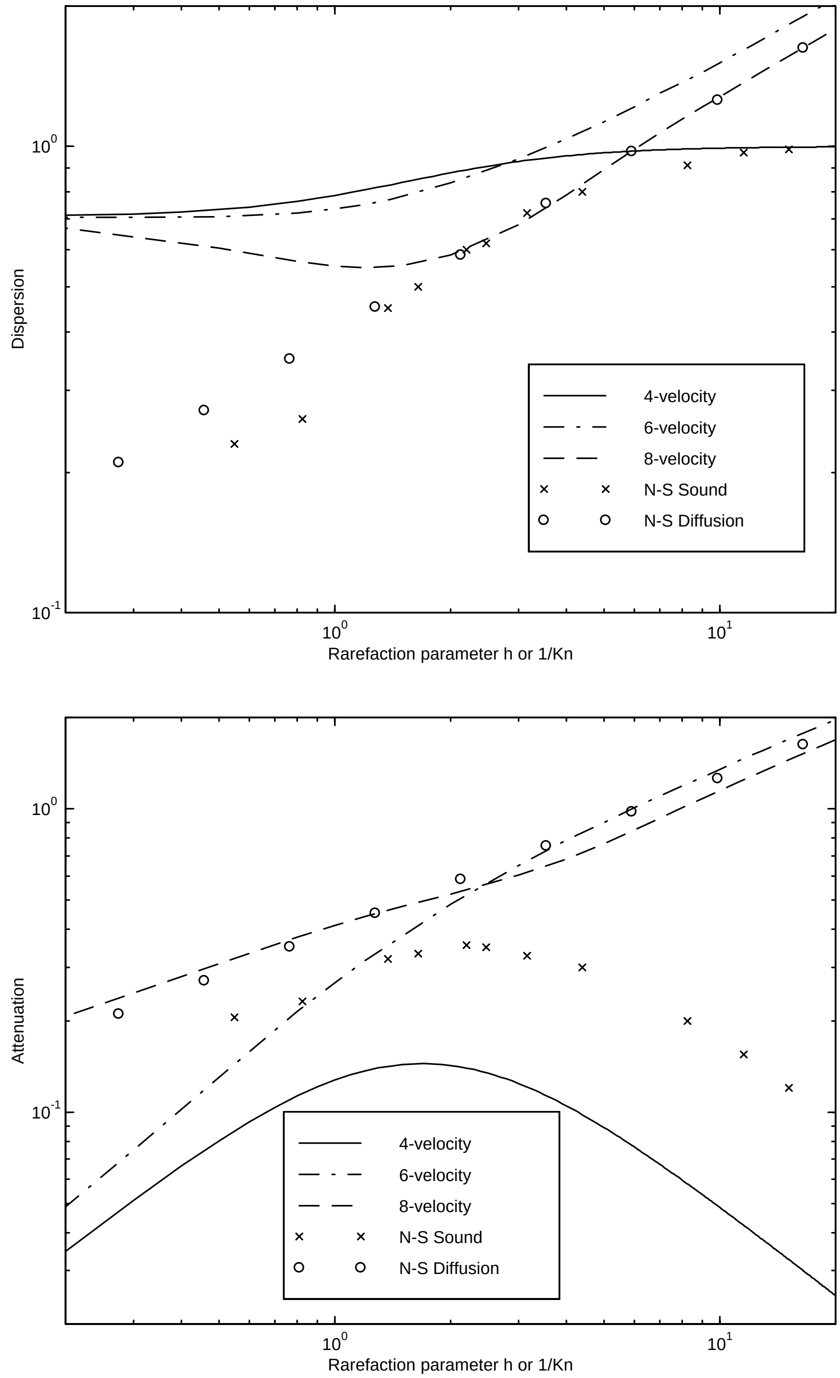

Fig. 1. Comparison with Navier-Stokes approach for dispersion (upper data) and attenuation (lower data) w.r.t. rarefaction parameter h.

may be due to the different Maxwellian (thermodynamic equilibrium state) corresponding to physical and/or unphysical invariants [13,16].

In brief summary, the dispersion $(k_r c/(\sqrt{2}\omega))$ reaches a continuum-value of one for the 4-velocity model once h increases to infinity. The attenuation $(k_i c/(\sqrt{2}\omega))$ for the same model, instead, firstly increases up to $h \sim 1.7$ then starts to decrease as h increases furthermore. As for the 6- and 8-velocity models, the attenuation and dispersion keep increasing without bound as h increases. The results may show the intrinsic thermodynamic properties of the Maxwellian states corresponding to the different models. By the situation $h \sim 5$, the diffusion mode having an attenuation only about twice that of the sound mode, this is perhaps the point at which some attention should be paid to the boundary conditions.

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